A geometric form of the Hahn-Banach extension theorem for L^0- linear functions and the Goldstine-Weston theorem in random normed modules

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Abstract In this paper, we present a geometric form of the Hahn-Banach extension theorem for L^0-linear functions and prove that the geometric form is equivalent to the analytic form of the Hahn-Banach extension theorem. Further, we use the geometric form to give a new proof of a known basic strict separation theorem in random locally convex modules. Finally, using the basic strict separation theorem we establish the Goldstine-Weston theorem in random normed modules under the two kinds of topologies—-the $(\varepsilon, \lambda)-$topology and the locally L^0-convex topology, and also provide a counterexample showing that the Goldstine-Weston theorem under the locally L^0-convex topology can only hold for random normed modules with the countable concatenation property.

Keywords: Hahn-Banach extension theorem, random locally convex module, random normed module, $(\epsilon, \lambda)-$topology, locally L^0-convex topology, separation theorem, Goldstine-Weston theorem

MSC(2000): 46A22, 46A16, 46H25, 46H05

1 Introduction

It is well known that the classical Hahn-Banach extension theorem for linear functionals has both its algebraic form and geometric form. The corresponding algebraic form of the Hahn-Banach extension theorem for random linear functionals are due to Guo in [1, 2]. The Hahn-Banach extension theorem for L^0-linear functions, namely Proposition 1.1 below, is due to [3, 4], an extremely simple proof of which was given in [5].

Before giving Proposition 1.1, we first recall some notation and terminology.

In the sequel of this paper, $(\Omega, \mathcal{F}, P)$ denotes a probability space, N the set of all positive integers, K the real number field R or the complex number field C, $\bar{R} = [-\infty, +\infty]$, $\bar{L}^0(\mathcal{F}, R)$ the set of equivalence classes of extended real-valued random variables on $(\Omega, \mathcal{F}, P)$, $L^0(\mathcal{F}, K)$

☆Supported by the National Natural Science Foundation of China (No. 10871016).

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the algebra of equivalence classes of $K-$valued random variables on $(\Omega, \mathcal{F}, P)$ under the ordinary scalar multiplication, addition and multiplication operations on equivalence classes, the null and unit elements are still denoted by 0 and 1, respectively.

It is well known from [6] that $\bar{L}^0(\mathcal{F}, R)$ is a complete lattice under the ordering $\leqslant$: $\xi \leqslant \eta$ iff $\xi^0(\omega) \leqslant \eta^0(\omega)$, for almost all ω in Ω (briefly, a.s.), where ξ^0 and η^0 are arbitrarily chosen representatives of ξ and η, respectively (see also Proposition 2.1 below). Furthermore, every subset G of $\bar{L}^0(\mathcal{F}, R)$ has a supremum and an infimum, denoted by $\bigvee G$ and $\bigwedge G$, respectively. In particular, $L^0(\mathcal{F}, R)$, as a sublattice of $\bar{L}^0(\mathcal{F}, R)$, is also a complete lattice in the sense that every subset with an upper bound has a supremum.

Specially, $L^0_+ = \{\xi \in L^0(\mathcal{F}, R) \mid \xi \geqslant 0\}$, $L^0_{++} = \{\xi \in L^0(\mathcal{F}, R) \mid \xi > 0 \text{ on } \Omega\}$, where for $A \in \mathcal{F}$, "$\xi > \eta$" on A means $\xi^0(\omega) > \eta^0(\omega)$ a.s. on A for any chosen representatives ξ^0 and η^0 of ξ and η, respectively. As usual, $\xi > \eta$ means $\xi \geqslant \eta$ and $\xi \neq \eta$.

Given a random locally convex module $(E, \mathcal{P})$ over K with base $(\Omega, \mathcal{F}, P)$, let $\mathcal{T}_{\varepsilon,\lambda}$ and $\mathcal{T}_c$ denote the $(\varepsilon, \lambda)-$topology and the locally L^0-convex topology for E, respectively, see [5, 7] and also Section 2 for the definitions of these two kinds of topologies.

Proposition 1.1 (The algebraic form of Hahn-Banach theorem for L^0-linear functions [3, 4, 7]). Let E be a left module over the algebra $L^0(\mathcal{F}, R)$, M an $L^0(\mathcal{F}, R)-$submodule in E, $g : M \to L^0(\mathcal{F}, R)$ an L^0-linear functional and $p : E \to L^0(\mathcal{F}, R)$ an L^0-sublinear functional such that $g(x) \leqslant p(x), \forall x \in M$. Then there exists an L^0-linear functional $f : E \to L^0(\mathcal{F}, R)$ such that f extends g and $f(x) \leqslant p(x), \forall x \in E$.

In this paper we present the following geometric form of Proposition 1.1, namely Proposition 1.2 below, and point out that the geometric form is equivalent to the algebraic form stated above.

Proposition 1.2(The geometric form of Hahn-Banach theorem for L^0-linear functions). Let E be a left module over the algebra $L^0(\mathcal{F}, R)$, M an $L^0(\mathcal{F}, R)-$submodule in E and G an L^0-convex and L^0-absorbent subset of E. If $g : M \to L^0(\mathcal{F}, R)$ is an L^0-linear functional and $g(y) \leqslant 1$ for any $y \in M \cap G$, then there exists an L^0-linear functional $f : E \to L^0(\mathcal{F}, R)$ such that f extends g and $f(x) \leqslant 1, \forall x \in G$.

In addition, we make use of the geometric form to give a new proof of the following known basic strict separation theorem in random locally convex modules:

Proposition 1.3 ([8]). *Let $(E, \mathcal{P})$ be a random locally convex module over K with base $(\Omega, \mathcal{F}, P)$, G a $\mathcal{T}_{\varepsilon,\lambda}-$closed and L^0-convex subset of E, $x_0 \in E \backslash G$, $\xi_{\mathcal{Q}} = \bigwedge\{\|x_0 - h\|_{\mathcal{Q}} \mid h \in G\}$ for each $\mathcal{Q} \in \mathcal{F}(\mathcal{P})$ and $\xi = \bigvee\{\xi_{\mathcal{Q}} \mid \mathcal{Q} \in \mathcal{F}(\mathcal{P})\}$. Then there exists a continuous module*

homomorphism f from $(E,\mathcal{T}_{\epsilon,\lambda})$ to $(L^0(\mathcal{F},K),\mathcal{T}_{\epsilon,\lambda})$ such that

$$(Ref)(x_0) > \bigvee\{(Ref)(y) \mid y \in G\},$$

where Ref denotes the real part of f, namely $f(x) = (Ref)(x) - i(Ref)(ix), \forall x \in E$ and

$$(Ref)(x_0) > \bigvee\{(Ref)(y) \mid y \in G\} \text{ on } [\xi > 0].$$

In the final part of this paper, we establish the Goldstine-Weston theorem in random normed modules under the two kinds of topologies, namely the $(\varepsilon,\lambda)-$topology and the locally L^0-convex topology, which are stated as follows:

Theorem 1.1. *Let $(E,\|\cdot\|)$ be an RN module over K with base $(\Omega,\mathcal{F},P)$, J the natural embedding mapping: $E\to E^{**}$ defined by $J(x)(g) = g(x)$ for any $x\in E$ and $g\in E^*$, $E(1) = \{x\in E \mid \|x\|\leqslant 1\}$ and $\overline{J(E(1))}^{w^*}_{\varepsilon,\lambda}$ the closure of $J(E(1))$ with respect to $\sigma_{\varepsilon,\lambda}(E^{**},E^*)$. Then $\overline{J(E(1))}^{w^*}_{\varepsilon,\lambda} = E^{**}(1)$, where $E^{**}(1) = \{\phi\in E^{**} \mid \|\varphi\|^{**}\leqslant 1\}$.*

Theorem 1.2. *Let $(E,\|\cdot\|)$ be an RN module over K with base $(\Omega,\mathcal{F},P)$ such that E has the countable concatenation property, J and $E(1)$ the same as in Theorem 1.1, and $\overline{J(E(1))}^{w^*}_{c}$ the closure of $J(E(1))$ with respect to $\sigma_c(E^{**},E^*)$. Then $\overline{J(E(1))}^{w^*}_{c} = E^{**}(1)$.*

Further, we give an example to show that $J(E(1))$ may not be dense in $E^{**}(1)$ under $\sigma_c(E^{**},E^*)$ if $(E,\|\cdot\|)$ has not the countable concatenation property.

The remainder of this paper is organized as follows: in Section 2 we will recapitulate some known basic facts, in Section 3 we will prove that the geometric form of Hahn-Banach extension theorem for L^0-linear functions is equivalent to the algebraic form and in Section 4 we will prove the Goldstine-Weston theorem in random normed modules.

2 Preliminaries

Proposition 2.1 ([6]). *For every subset G of $\bar{L}^0(\mathcal{F},R)$ there exist countable subsets $\{a_n \mid n\in N\}$ and $\{b_n \mid n\in N\}$ of G such that $\bigvee G = \bigvee_{n\geqslant 1} a_n$ and $\bigwedge G = \bigwedge_{n\geqslant 1} b_n$. Further, if G is directed (dually directed) with respect to $\leqslant$, then the above $\{a_n \mid n\in N\}$ (accordingly, $\{b_n \mid n\in N\}$) can be chosen as nondecreasing (correspondingly, nonincreasing) with respect to $\leqslant$.*

For an arbitrarily chosen representative ξ^0 of $\xi\in L^0(\mathcal{F},K)$, define the two random variables $(\xi^0)^{-1}$ and $|\xi^0|$ by $(\xi^0)^{-1}(\omega) = 1/\xi^0(\omega)$ if $\xi^0(\omega)\neq 0$, and $(\xi^0)^{-1}(\omega) = 0$ otherwise, and by $|\xi^0|(\omega) = |\xi^0(\omega)|$, $\forall\omega\in\Omega$. Then the equivalent class $Q(\xi)$ of $(\xi^0)^{-1}$ is called the generalized inverse of ξ and the equivalent class $|\xi|$ of $|\xi^0|$ the absolute value of ξ.

Besides, for any $A \in \mathcal{F}$, A^c denotes the complement in Ω, $\tilde{A} := \{B \in \mathcal{F} \mid P(A\Delta B) = 0\}$ the equivalence class of A, where Δ is the symmetric difference operation, I_A the characteristic function of A and $\tilde{I}_A$ the equivalence class of I_A. Given two ξ and η in $L^0(\mathcal{F}, R)$, and $A = \{\omega \in \Omega \mid \xi^0 \neq \eta^0\}$, where ξ^0 and η^0 are arbitrarily chosen representatives of ξ and η respectively, then we always write $[\xi \neq \eta]$ for the equivalence class of A and $I_{[\xi \neq \eta]}$ for $\tilde{I}_A$, one can also understand the implication of such notations as $I_{[\xi \leqslant \eta]}$, $I_{[\xi < \eta]}$ and $I_{[\xi = \eta]}$.

Definition 2.1 ([9, 10]). (1) *Let E be a linear space over K, then a mapping $f : E \to L^0(\mathcal{F}, K)$ is called a random linear functional on E if f is linear;*

(2) *If E is a linear space over R, then a mapping $f : E \to L^0(\mathcal{F}, R)$ is called a random sublinear functional on E if $f(\alpha x) = \alpha \cdot f(x)$ for any positive real number α and $x \in E$, and if $f(x + y) \leqslant f(x) + f(y), \forall x, y \in E$;*

(3) *Let E be a linear space over K, then a mapping $f : E \to L^0_+$ is called a random seminorm on E if $f(\alpha x) = |\alpha| \cdot f(x), \forall \alpha \in K$ and $x \in E$, and if $f(x + y) \leqslant f(x) + f(y), \forall x, y \in E$;*

(4) *Let E be a left module over the algebra $L^0(\mathcal{F}, K)$, then a mapping $f : E \to L^0(\mathcal{F}, K)$ is called a L^0-linear function on E if f is a module homomorphism;*

(5) *Let E be a left module over the algebra $L^0(\mathcal{F}, R)$, a mapping $f : E \to L^0(\mathcal{F}, R)$ is called an L^0-sublinear functional on E if f is a random sublinear function on E such that $f(\xi \cdot x) = \xi \cdot f(x), \forall \xi \in L^0_+$ and $x \in E$;*

(6) *Let E be a left module over the algebra $L^0(\mathcal{F}, K)$, then a mapping $f : E \to L^0_+$ is called an L^0-seminorm on E if f is a random seminorm on E such that $f(\xi \cdot x) = |\xi| \cdot f(x), \forall \xi \in L^0(\mathcal{F}, K)$ and $x \in E$.*

Definition 2.2 ([5, 9, 11]). *An ordered pair $(E, \mathcal{P})$ is called a random locally convex space over K with base $(\Omega, \mathcal{F}, P)$ if E is a linear space over K and $\mathcal{P}$ is a family of random seminorms on E such that the following axiom is satisfied:*

(1) *$\bigvee\{\|x\| \mid \|\cdot\| \in \mathcal{P}\} = 0$ implies $x = \theta$ (the null element of E).*
In addition, if E is a left module over the algebra $L^0(\mathcal{F}, K)$ and each $\|\cdot\|$ in $\mathcal{P}$ is an L^0-seminorm, then such a random locally convex space is called a random locally convex module.

Remark 2.1. Let $(E, \mathcal{P})$ be a random locally convex space (a random locally convex module) over K with base $(\Omega, \mathcal{F}, P)$. If $\mathcal{P}$ degenerates to a singleton $\{\|\cdot\|\}$, then $(E, \|\cdot\|)$ is exactly a random normed space (briefly, an RN space) (correspondingly, a random normed module (briefly, an RN module)). Specially, $(L^0(\mathcal{F}, K), |\cdot|)$ is an RN module.

In the sequel, for a random locally convex space $(E, \mathcal{P})$ with base $(\Omega, \mathcal{F}, P)$ and for each

finite subfamily $\mathcal{Q}$ of $\mathcal{P}$, $\|\cdot\|_{\mathcal{Q}}: E \to L^0_+(\mathcal{F})$ always denotes the random seminorm of E defined by $\|x\|_{\mathcal{Q}} = \bigvee\{\|x\| \mid \|\cdot\| \in \mathcal{Q}\}, \forall x \in E$, and $\mathcal{F}(\mathcal{P})$ the set of finite subfamilies of $\mathcal{P}$.

For each random locally convex space $(E, \mathcal{P})$ over K with base $(\Omega, \mathcal{F}, P)$, $\mathcal{P}$ can induce the following two kinds of topologies, namely the $(\varepsilon, \lambda)-$topology and the locally L^0-convex topology.

Definition 2.3 ([5, 9, 11]). *Let $(E, \mathcal{P})$ be a random locally convex space over K with base $(\Omega, \mathcal{F}, P)$. For any positive real numbers ε and λ such that $0 < \lambda < 1$, and any $\mathcal{Q} \in \mathcal{F}(\mathcal{P})$, let $N_\theta(\mathcal{Q}, \varepsilon, \lambda) = \{x \in E \mid P\{\omega \in \Omega \mid \|x\|_{\mathcal{Q}}(\omega) < \varepsilon\} > 1 - \lambda\}$, then $\{N_\theta(\mathcal{Q}, \varepsilon, \lambda) \mid \mathcal{Q} \in \mathcal{F}(\mathcal{P}), \varepsilon > 0, 0 < \lambda < 1\}$ is easily verified to be a local base at the null vector θ of some Hausdorff linear topology, called the $(\varepsilon, \lambda)-$topology for E induced by $\mathcal{P}$.*

From now on, the $(\varepsilon, \lambda)-$topology for each random locally convex space is always denoted by $\mathcal{T}_{\varepsilon,\lambda}$ when no confusion occurs.

Definition 2.4([7, 11]). *Let $(E, \mathcal{P})$ be a random locally convex space over K with base $(\Omega, \mathcal{F}, P)$. For any $\mathcal{Q} \in \mathcal{F}(\mathcal{P})$ and $\varepsilon \in L^0_{++}$, let $N_\theta(\mathcal{Q}, \varepsilon) = \{x \in E \mid \|x\|_{\mathcal{Q}} \leqslant \varepsilon\}$. A subset G of E is called $\mathcal{T}_c-$open if for each $x \in G$ there exists some $N_\theta(\mathcal{Q}, \varepsilon)$ such that $x + N_\theta(\mathcal{Q}, \varepsilon) \subset G$, $\mathcal{T}_c$ denotes the family of $\mathcal{T}_c-$open subsets of E. Then it is easy to see that $(E, \mathcal{T}_c)$ is a Hausdorff topological group with respect to the addition on E. $\mathcal{T}_c$ is called the locally L^0-convex topology for E induced by $\mathcal{P}$.*

From now on, the locally L^0-convex topology for each random locally convex space is always denoted by $\mathcal{T}_c$ when no confusion occurs.

Now, we present the definition of random conjugate spaces of a random locally convex space. Historically, the earliest two notions of a random conjugate space of a random locally convex space were introduced in [9, 12], respectively. As shown in [5, 11], it turned out that they just correspond to the $(\varepsilon, \lambda)-$topology and the locally L^0-convex topology in the context of a random locally convex module, respectively!

Definition 2.5([12]). *Let $(E, \mathcal{P})$ be a random locally convex space over K with base $(\Omega, \mathcal{F}, P)$. A random linear functional $f : E \to L^0(\mathcal{F}, K)$ is called an a.s. bounded random linear functional of type I if there are some $\xi \in L^0_+$ and $\mathcal{Q} \in \mathcal{F}(\mathcal{P})$ such that $|f(x)| \leqslant \xi \cdot \|x\|_{\mathcal{Q}}, \forall x \in E$. Denote by E^*_I the set of a.s. bounded random linear functional of type I on E. The module multiplication operation $\cdot : L^0(\mathcal{F}, K) \times E^*_I \to E^*_I$ is defined by $(\xi f)(x) = \xi(f(x)), \forall \xi \in L^0(\mathcal{F}, K), f \in E^*_I$ and $x \in E$. It is easy to see that E^*_I is a left module over $L^0(\mathcal{F}, K)$, called the random conjugate space of type I of E.*

Definition 2.6([10]). *Let $(E, \mathcal{P})$ be a random locally convex space over K with base $(\Omega, \mathcal{F}, P)$. A random linear functional $f : E \to L^0(\mathcal{F}, K)$ is called an a.s. bounded random linear functional of type II on E if there exist a countable partition $\{A_i \mid i \in N\}$ of Ω to $\mathcal{F}$, a sequence $\{\xi_i \mid i \in N\}$ in L^0_+ and a sequence $\{\mathcal{Q}_i \mid i \in N\}$ in $\mathcal{F}(\mathcal{P})$ such that $|f(x)| \leqslant \Sigma_{i=1}^{\infty} \tilde{I}_{A_i} \cdot \xi_i \cdot \|x\|_{\mathcal{Q}_i}, \forall x \in E$. Denote by E^*_{II} the $L^0(\mathcal{F}, K)-$module of a.s. bounded random linear functional of type II on E, called the random conjugate space of type II of E.*

Definition 2.7. *Let $(E, \mathcal{P})$ be a random locally convex module over K with base $(\Omega, \mathcal{F}, P)$ and define $E^*_{\varepsilon,\lambda}$, E^*_c as follows:*

(1) $E^*_{\varepsilon,\lambda} = \{f \mid f \text{ is a continuous module homomorphism from } (E, \mathcal{T}_{\varepsilon,\lambda}) \text{ to } (L^0(\mathcal{F}, K), \mathcal{T}_{\varepsilon,\lambda})\}$,

(2) $E^*_c = \{f \mid f \text{ is a continuous module homomorphism from } (E, \mathcal{T}_c) \text{ to } (L^0(\mathcal{F}, K), \mathcal{T}_c)\}$.

Propositions 2.2 and 2.3 below give the topological characterizations of an element in E^*_I and E^*_{II}, respectively.

Proposition 2.2([5, 9]). *Let $(E, \mathcal{P})$ be a random locally convex module over K with base $(\Omega, \mathcal{F}, P)$ and $f : E \to L^0(\mathcal{F}, K)$ a random linear functional. Then $f \in E^*_I$ iff f is a continuous module homomorphism from $(E, \mathcal{T}_c)$ to $(L^0(\mathcal{F}, K), \mathcal{T}_c)$, namely $E^*_I = E^*_c$.*

Proposition 2.3([5, 13]). *Let $(E, \mathcal{P})$ be a random locally convex module over K with base $(\Omega, \mathcal{F}, P)$ and $f : E \to L^0(\mathcal{F}, K)$ a random linear functional. Then $f \in E^*_{II}$ iff f is a continuous module homomorphism from $(E, \mathcal{T}_{\epsilon,\lambda})$ to $(L^0(\mathcal{F}, K), \mathcal{T}_{\epsilon,\lambda})$, namely $E^*_{II} = E^*_{\varepsilon,\lambda}$.*

Remark 2.2. It is clear that $E^*_c \subset E^*_{\varepsilon,\lambda}$ from Proposition 2.2 and 2.3. Specially, if $(E, \|\cdot\|)$ is an RN module over K with base $(\Omega, \mathcal{F}, P)$, then $E^*_c = E^*_{\varepsilon,\lambda}$ (see [5] for details), in which case we denote $E^*_{\varepsilon,\lambda}$ or E^*_c by E^*, further define $\|\cdot\|^* : E^* \to L^0_+$ by $\|f\|^* = \bigvee\{|f(y)| \mid y \in E \textit{ and } \|y\| \leqslant 1\}$ and $\cdot : L^0(\mathcal{F}, K) \times E^* \to E^*$ by $(\xi \cdot f)(x) = \xi \cdot (f(x))$ for any $\xi \in L^0(\mathcal{F}, K)$, $f \in E^*$ and $x \in E$. Then it is clear that $(E^*, \|\cdot\|^*)$ is an RN module over K with base $(\Omega, \mathcal{F}, P)$, called the random conjugate space of $(E, \|\cdot\|)$ (see [14]).

The following notion of a gauge function was presented by D.Filipović, M.Kupper and N.Vogelpoth in [7] for the first time.

Definition 2.8 ([5, 7]). *Let E be a left module over the algebra $L^0(\mathcal{F}, K)$ and A a subset of E. Then*

(1) *A is called L^0-convex if $\xi \cdot x + \eta \cdot y \in E$ for any x and y in A and for any ξ and η in L^0_+ such that $\xi + \eta = 1$;*

(2) *A is called L^0-absorbent if for each $x \in E$ there exists some $\xi \in L^0_{++}$ such that $x \in \xi \cdot A := \{\xi \cdot a \mid a \in A\}$;*

(3) *A is called L^0-balanced if $\xi \cdot x \in A$ for any $x \in A$ and $\xi \in L^0(\mathcal{F}, K)$ such that $|\xi| \leqslant 1$.*

Definition 2.9 ([7]). *Let E be a left module over $L^0(\mathcal{F}, K)$. Then the gauge function $p_G: E \to \bar{L}^0_+$ of a set $G \subset E$ is defined by*

$$p_G(x) := \bigwedge\{\xi \in L^0_+ \mid x \in \xi \cdot G\}.$$

Proposition 2.4 ([7]). *Let E be a left module over $L^0(\mathcal{F}, K)$. The gauge function p_G of an L^0-absorbent set $G \subset E$ has the following properties:*

(i) $p_G(x) \leqslant 1$ for all $x \in G$;

(ii) $\tilde{I}_A \cdot p_G(\tilde{I}_A \cdot x) \leqslant \tilde{I}_A \cdot p_G(x)$ for all $x \in E$ and $A \in \mathcal{F}$;

(iii) $\xi \cdot p_G(\tilde{I}_{[\xi>0]} \cdot x) = p_G(\xi \cdot x)$ for all $x \in E$ and $\xi \in L^0_+$; in particular, $\xi \cdot p_G(x) = p_G(\xi \cdot x)$ if $\xi \in L^0_{++}$.

A non-empty L^0-absorbent L^0-convex set $G \subset E$ always contains the origin; depending on the choice of $G \subset E$, the gauge function may be an L^0-sublinear or an L^0-seminorm.

Proposition 2.5 ([7]). *Let E be a left module over $L^0(\mathcal{F}, K)$. Then the gauge function p_G of an L^0-absorbent L^0-convex set $G \subset E$ satisfies:*

(i) $p_G(x) = \bigwedge\{\xi \in L^0_{++} \mid x \in \xi \cdot G\}$ for all $x \in E$;

(ii) $\xi \cdot p_G(x) = p_G(\xi \cdot x)$ for all $\xi \in L^0_+$ and $x \in E$;

(iii) $p_G(x + y) \leqslant p_G(x) + p_G(y)$ for all $x,\ y \in E$;

(iv) for all $x \in E$ there exists a sequence $\{\eta_n\}_{n=1}^{\infty}$ in L^0_{++} such that

$$\eta_n \searrow p_G(x) \quad a.s.,$$

in particular, p_G is an L^0-sublinear functional since $0 \in G$;

if G is also L^0-balanced, then p_G satisfies:

(v) $p_G(\xi \cdot x) = |\xi| \cdot p_G(x)$ for all $\xi \in L^0$ and for all $x \in E$, namely p_G is an L^0-seminorm.

Proposition 2.6 ([7]). *Let E be a left module over $L^0(\mathcal{F}, K)$. Then the gauge function p_G of an L^0-absorbent L^0-convex set $G \subset E$ satisfies that $p_G(x) \geqslant 1$ for all $x \in E$ with $\tilde{I}_A \cdot x \notin \tilde{I}_A \cdot G$ for all $A \in \mathcal{F}$ with $P(A) > 0$.*

3 The geometric form of Hahn-Banach extension theorem for L^0-linear functions

Theorem 3.1 Proposition 1.1 is equivalent to Proposition 1.2.

Proof. Let p_G be the gauge function of G, namely

$$p_G(x) = \bigwedge\{\xi \in L^0_{++} \mid \xi \cdot x \in G\},\ \forall x \in E.$$

Since G is an L^0-convex and L^0-absorbent subset of E, p_G is an L^0-sublinear functional on E by Proposition 2.5, and since $g(y) \leqslant 1$ for any $y \in M \cap G$, then for any $x \in M$ and $\lambda \in L^0_{++}$ we can obtain $g(x) \leqslant \lambda$ when $x \in \lambda \cdot G$, namely $g(x) \leqslant p_G(x)$. From Proposition 3.1, there exists an L^0-linear functional $f : E \to L^0(\mathcal{F}, R)$ such that f extends g and $f(x) \leqslant p_G(x), \forall x \in E$. Therefore, we have that

$$f(x) \leqslant 1, \forall x \in G.$$

Conversely, let $G = \{x \in E \mid p(x) \leqslant 1\}$, then it is clear that $g,\ M$ and G satisfy Proposition 1.2, hence there exists an L^0-linear functional $f : E \to L^0(\mathcal{F}, R)$ such that f extends g and

$$f(x) \leqslant 1, \forall x \in G$$

by Proposition 1.2, so that we can have that $f(x) \leqslant p(x), \forall x \in E$. □

If $K = C$, we have the following geometric form:

Theorem 3.2. Let E be a left module over the algebra $L^0(\mathcal{F}, C)$, M an $L^0(\mathcal{F}, C)-$submodule in E and G an L^0-convex and L^0-absorbent subset of E. If $g : M \to L^0(\mathcal{F}, C)$ is an L^0-linear functional and $(Reg)(y) \leqslant 1$ for any $y \in M \cap G$, then there exists an L^0-linear functional $f : E \to L^0(\mathcal{F}, C)$ such that f extends g and $(Ref)(x) \leqslant 1, \forall x \in G$.

Now, we give a new proof of Proposition 1.3 by the geometric form of Hahn-Banach extension theorem for L^0-functions. In fact, we need only to prove the following basic strict separation theorem for the case of RN modules, namely Proposition 3.1 below, since by Proposition 3.1 one can easily complete the remaining part of the proof of Proposition 1.3, see [8] for details.

Let $(E, \|\cdot\|)$ be an RN module, G a $\mathcal{T}_{\varepsilon,\lambda}-$closed L^0-convex subset of E and $x_0 \in E \setminus G$. Then $\{\|x_0 - g\| \mid g \in G\}$ is a dually directed subset in L^0_+ and one can obtain the following claim: there exists $A \in \mathcal{F}$ with $P(A) > 0$ such that $\bigwedge\{\|x_0 - g\| \mid g \in G\} > 0$ on A from Lemma 3.8 in [5].

Proposition 3.1 ([8]). *Let $(E, \|\cdot\|)$ be an RN module over R with base $(\Omega, \mathcal{F}, P)$, G a $\mathcal{T}_{\varepsilon,\lambda}-$closed and L^0-convex subset of E, $x_0 \in E \setminus G$, and $\xi = \bigwedge\{\|x_0 - h\| \mid h \in G\}$. Then there*

exists a continuous module homomorphism f from $(E, \mathcal{T}_{\epsilon,\lambda})$ to $(L^0(\mathcal{F}, R), \mathcal{T}_{\epsilon,\lambda})$ such that

$$f(x_0) > \bigvee\{|f(y)| \mid y \in G\}$$

and

$$f(x_0) > \bigvee\{|f(y)| \mid y \in G\} \text{ on } [\xi > 0].$$

Proof. Without loss of generality, we can assume $\theta \in G$ (otherwise, by a translation). It is easy to see that $\tilde{I}_B \cdot x_0 \notin G$ for any $B \in \mathcal{F}$ with $B \subset A$ and $P(B) > 0$. In fact, assume that $\tilde{I}_B \cdot x_0 \in G$, then $\xi = 0$ on B, which is contradict to $\tilde{A} = [\xi > 0]$ and $B \subset A$. Let $M = \{x \in E \mid \|x\| \leqslant \frac{1}{3}\tilde{I}_A \cdot \xi \text{ on } A\}$, then it is clear that M is L^0-convex and L^0-absorbent. Further, let $G + M = \{h + x \mid h \in G, x \in M\}$, then $G + M$ is also an L^0-convex and L^0-absorbent subset of E. Since $\theta \in G + M$ and $G + M$ is an L^0-convex, we have that $\tilde{I}_F \cdot (G + M) \subset G + M$ for every subset $F \in \mathcal{F}$ with $P(F) > 0$.

Let p_{G+M} be the gauge function of $G + M$, then p_{G+M} is an L^0-sublinear functional on E by Proposition 2.5. It is easy to see that $p_{G+M}(x) = 0$ on A^c for any $x \in E$. Now we prove that $p_{G+M}(x_0) > 1$ on A. In fact, for any $z = z_G + z_M$, where $z_G \in G$ and $z_M \in M$, since $\|(x_0 - z)\| \geqslant \|x_0 - z_G\| - \|z_M\|$, we can obtain that $\|(x_0 - z)\| \geqslant \frac{2}{3}\xi > 0$ on A from $\|x_0 - z_G\| \geqslant \xi$ and $\|z_M\| \leqslant \frac{1}{3}\cdot\xi$ on A. Thus $x_0 \notin G+M$ and $p_{G+M}(x_0) \neq 0$ by Definition 2.8. From Proposition 2.5, there exists a sequence $\{\eta_n \mid n \in N\} \subset L^0_{++}$ such that $x_0 \in \eta_n\cdot(G+M)$ and $\eta_n \searrow p_{G+M}(x_0)$. According to $x_0 \in E \setminus (G + M)$ and $\bigwedge\{\|x_0 - h\| \mid h \in G + M\} > \frac{1}{3}\xi$ on A, we have that $\eta_n > 1$ on A for any $n \in N$ and hence $p_{G+M}(x_0) \geqslant 1$ on A. Let $\tilde{D} = [p_{G+M}(x_0) = 1] \cap \tilde{A}$, then we will prove that $P(D) > 0$ is impossible: if $P(D > 0)$, it is clear that $\eta_n \searrow \tilde{I}_D$ and $Q(\eta_n) \nearrow \tilde{I}_D$ on D; since $x_0 \in \eta_n \cdot (G + M)$ and $\eta_n > 1$ on D, we have $Q(\eta_n) \cdot x_0 \in (Q(\eta_n) \cdot \eta_n) \cdot (G + M) \subset G + M$ and $\|x_0 - Q(\eta_n) \cdot x_0\| = \|(1 - Q(\eta_n)) \cdot x_0\| \searrow 0$ on D, which contradicts to the fact that $\bigwedge\{\|x_0 - h\| \mid h \in G + M\} > \frac{1}{3}\xi$ on A. Hence $P(D) = 0$ and $p_{G+M}(x_0) > 1$ on A.

Now we prove that the gauge function p_{G+M} of $G+M$ is continuous under the $(\epsilon, \lambda)-$topology. Let x be an arbitrary element of E, $\tilde{H}_x = [\|x\| \neq 0]$ and $\bar{t} = \frac{1}{3}\tilde{I}_{A\cap H_x} \cdot \xi \cdot \|x\|^{-1}$, then $\bar{t} \cdot x \in M \subset G + M$. Thus we have

$$p_{G+M}(x) = 0$$

on H_x^c and

$$p_{G+M}(x) \leqslant 3\tilde{I}_{A\cap H_x} \cdot Q(\xi) \cdot \|x\|$$

on H_x.

Therefore we can obtain that

$$p_{G+M}(x) \leqslant (3Q(\xi)+1)\cdot \|x\|$$

and p_{G+M} is continuous under the $(\epsilon,\lambda)-$topology.

Let $U=\{k\cdot x_0 \mid k\in L^0(\mathcal{F},R)\}$, then U is an L^0-submodule of E. Define an L^0-linear function $g: U\to L^0(\mathcal{F},R)$, where $g(x_0)=\tilde{I}_A\cdot p_{G+M}(x_0)$. Then we have that $g(y)\leqslant 1,\ \forall y\in U\cap(G+M)$. By Proposition 1.2, there exists an L^0-linear function $\bar{g}: E\to L^0(\mathcal{F},R)$ such that $\bar{g}$ extends g and $\bar{g}(x)\leqslant 1, \forall y\in G+M$. Hence, $\bar{g}(x_0)>1$ on A and

$$\bar{g}(x_0)>\bigvee\{|f(y)| \mid y\in G\} \ on\ A.$$

Let $f=\tilde{I}_A\cdot\bar{g}$, since $f\leqslant p_{G+M}$ from Remark 3.1, it is easy to see that $f\in E^*_{\varepsilon,\lambda}$ and $f(x)=0$ on A^c. Therefore, it is clear that

$$f(x_0)>\bigvee\{|f(y)| \mid y\in G\}$$

and

$$f(x_0)>\bigvee\{|f(y)| \mid y\in G\} \ on\ A. \quad \square$$

4 The Goldstine-Weston theorem in RN modules

Before giving the proofs of Theorem 1.1 and 1.2, we first present some necessary definitions and lemmas.

Definition 4.1 ([5, 11]). *Let E be a left module over the algebra $L^0(\mathcal{F},K)$. Such a formal sum $\sum_{n\geqslant 1}\tilde{I}_{A_n}x_n$ for some countable partition $\{A_n, n\in N\}$ of Ω to $\mathcal{F}$ and some sequence $\{x_n \mid n\in N\}$ in E, is called a countable concatenation of $\{x_n \mid n\in N\}$ with respect to $\{A_n, n\in N\}$. Furthermore a countable concatenation $\sum_{n\geqslant 1}\tilde{I}_{A_n}x_n$ is well defined or $\sum_{n\geqslant 1}\tilde{I}_{A_n}x_n\in E$ if there is $x\in E$ such that $\tilde{I}_{A_n}x=\tilde{I}_{A_n}x_n,\ \forall n\in N$. A subset G of E is called having the countable concatenation property if every countable concatenation $\sum_{n\geqslant 1}\tilde{I}_{A_n}x_n$ with $x_n\in G$ for each $n\in N$ still belongs to G, namely $\sum_{n\geqslant 1}\tilde{I}_{A_n}x_n$ is well defined and there exists $x\in G$such that $x=\sum_{n\geqslant 1}\tilde{I}_{A_n}x_n$.*

Lemma 4.1 ([5]). *Let $(E,\mathcal{P})$ be a random locally convex module over K with base $(\Omega,\mathcal{F},P)$, $G\subset E$ a subset having the countable concatenation property. Then $\bar{G}_{\varepsilon,\lambda}=\bar{G}_c$.*

Now, let us recall the random weak topology and the random weak star topology.

Definition 4.2 ([10, 11]). *Let $(E,\|\cdot\|)$ be an RN module over K with base $(\Omega,\mathcal{F},P)$, $(E^*,\|\cdot\|^*)$ the random conjugate space of E. For any $f\in E^*$, define $\|\cdot\|_f: E\to L^0_+$ by $\|x\|_f=$*

$|f(x)|,\ \forall x \in E,$ *and denote* $\{\|\cdot\|_f \mid f \in E^*\}$ *by* $\sigma(E,E^*)$*, it is clear that* $(E,\sigma(E,E^*))$ *is a random locally convex module over* K *with base* $(\Omega,\mathcal{F},P)$*. Then the* $(\varepsilon,\lambda)-$*topology* $\sigma_{\varepsilon,\lambda}(E,E^*)$ *and the locally* L^0-*convex topology* $\sigma_c(E,E^*)$ *on* E *induced by* $\sigma(E,E^*)$ *are called random weak* $(\varepsilon,\lambda)-$*topology and random weak locally* L^0-*convex topology on* E*, respectively.*

Remark 4.1. Similarly, we can define the random weak star $(\varepsilon,\lambda)-$topology $\sigma_{\varepsilon,\lambda}(E^*,E)$ and the random weak star locally L^0-convex topology $\sigma_c(E^*,E)$ on E^*, respectively.

Lemma 4.2 ([10]). *Let* $(E,\|\cdot\|)$ *be an RN module over* K *with base* $(\Omega,\mathcal{F},P)$*, then* $(E^*,\sigma_c(E^*,E))^* = E$*. Furthermore, if* E *has the countable concatenation property, then* $(E^*,\sigma_{\varepsilon,\lambda}(E^*,E))^* = E$*.*

If $(B,\|\cdot\|)$ is a normed space and $(B',\|\cdot\|')$ is the classical conjugate space of B, we have that $B'(1)=\{f\in B' \mid \|f\|'\leqslant 1\}$ is compact under the weak star topology by the well known Banach-Alaoglu theorem. Hence, $B'(1)$ is closed with respect to the weak star topology. Let $(E,\|\cdot\|)$ be an *RN* module over K with base $(\Omega,\mathcal{F},P)$ and $(E^*,\|\cdot\|^*)$ the random conjugate space. In [15], Guo proved that $E^*(1)=\{f\in E^* \mid \|f\|^*\leqslant 1\}$ is not compact under $\sigma_c(E^*,E)$ unless $(\Omega,\mathcal{F},P)$ is essentially purely $P-$atomic. But Lemma 4.3 below indicates that $E^*(1)$ is still closed with respect to both $\sigma_{\varepsilon,\lambda}(E^*,E)$ and $\sigma_c(E^*,E)$.

Lemma 4.3. *Let* $(E,\|\cdot\|)$ *be an RN module over* K *with base* $(\Omega,\mathcal{F},P)$ *such that* E *has the countable concatenation property. Then* $E^*(1)=\{f\in E^* \mid \|f\|^*\leqslant 1\}$ *is closed with respect to both* $\sigma_{\varepsilon,\lambda}(E^*,E)$ *and* $\sigma_c(E^*,E)$*.*

Proof. Since it is clear that $E^*(1)$ has the countable concatenation property, we need only to check that $E^*(1)$ is closed with respect to $\sigma_c(E^*,E)$. For any $f\in E^*\setminus E^*(1)$, there exists $A\in\mathcal{F}$ such that $P(A)>0$ and $\|f\|^*>1$ on A. From $\|f\|^*=\bigvee\{|f(x)| \mid \|x\|\leqslant 1\}$, there are $x_f\in E$, $\|x_f\|\leqslant 1$ and $B\in\mathcal{F}$, $B\subset A$ with $P(B)>0$ such that $|f(x_f)|>1$ on B. Let

$$\varepsilon = \tilde{I}_{B^c} + \frac{|f(x_f)|-1}{2}\cdot\tilde{I}_B$$

and

$$B(x_f,\varepsilon)=\{g\in E^* \mid |g(x_f)|\leqslant\varepsilon\},$$

then $B(x_f,\varepsilon)$ is a neighborhood of θ in E^* with respect to $\sigma_c(E^*,E)$ and, for any $h\in B(x_f,\varepsilon)$ it is easy to see that

$$|(f+h)(x_f)| \geqslant |f(x_f)|-|h(x_f)|$$

and

$$(f+h)(x_f)| \geqslant \frac{|f(x_f)|+1}{2} > 1$$

on B. Hence, $f+h \notin E^*(1)$, namely $f+B(x_f,\varepsilon) \subset E^* \setminus E^*(1)$. Consequently, $E^*(1)$ is closed with respect to $\sigma_c(E^*,E)$. □

Lemma 4.4 *Let $(E,\|\cdot\|)$ be an RN module over K with base $(\Omega,\mathcal{F},P)$ such that E has the countable concatenation property, J, $E(1)$, $J(E(1))$ and $\overline{J(E(1))}^{w^*}_{\varepsilon,\lambda}$ the same as in Theorem 1.1. Then $\overline{J(E(1))}^{w^*}_{\varepsilon,\lambda} = E^{**}(1)$.*

Proof. Since $E^{**}(1)=\{\varphi \in E^{**}(1) \mid \|\varphi\|^{**} \leqslant 1\}$ is closed with respect to $\sigma_{\varepsilon,\lambda}(E^{**},E^*)$ from Lemma 4.3, it follows that $\overline{J(E(1))}^{w^*}_{\varepsilon,\lambda} \subset E^{**}(1)$.

Now, we prove that $E^{**}(1) \subset \overline{J(E(1))}^{w^*}_{\varepsilon,\lambda}$. We only need to prove that for any $\psi \in E^{**} \setminus \overline{J(E(1))}^{w^*}_{\varepsilon,\lambda}$ there is $A_\psi \in \mathcal{F}$ such that $P(A_\psi)>0$ and $\|\psi\|^{**}>1$ on A_ψ. Since E has the countable concatenation property, we have that $(E^{**},\sigma_{\varepsilon,\lambda}(E^{**},E^*))^*=E^*$ by Lemma 4.2 and that there exists $\bar{f} \in E^*$ such that

$$(\mathrm{Re}\bar{f})(\psi) > \bigvee\{(\mathrm{Re}\bar{f})(g) \mid g \in \overline{J(E(1))}^{w^*}_{\varepsilon,\lambda}\}$$

and

$$(\mathrm{Re}\bar{f})(\psi) > \bigvee\{(\mathrm{Re}\bar{f})(g) \mid g \in \overline{J(E(1))}^{w^*}_{\varepsilon,\lambda}\} \text{ on } [\xi>0]$$

by Proposition 1.3, where ξ is the same as in Proposition 1.3. For any $y \in \overline{J(E(1))}^{w^*}_{\varepsilon,\lambda}$, it is easy to see that $||\bar{f}(y)|\cdot Q(\bar{f}(y))| \leqslant 1$ on Ω, $(|\bar{f}(y)|\cdot Q(\bar{f}(y)))\cdot y \in \overline{J(E(1))}^{w^*}_{\varepsilon,\lambda}$ and $\bar{f}((|\bar{f}(x)|\cdot Q(\bar{f}(y)))\cdot y)=|\bar{f}(y)|$. Hence, we have that

$$\bigvee\{(\mathrm{Re}\bar{f})(g) \mid g \in \overline{J(E(1))}^{w^*}_{\varepsilon,\lambda}\} = \bigvee\{(|\bar{f}(g)| \mid g \in \overline{J(E(1))}^{w^*}_{\varepsilon,\lambda}\}.$$

Let $f=Q(\|\bar{f}\|^*)\cdot\bar{f}$ and $A_\psi=[\xi>0]$, then we have that $\|f\|^*=1$ on A_ψ and

$$(\mathrm{Re}f)(\psi) > \bigvee\{(\mathrm{Re}f)(g) \mid g \in \overline{J(E(1))}^{w^*}_{\varepsilon,\lambda}\}$$

and

$$(\mathrm{Re}f)(\psi) > \bigvee\{(\mathrm{Re}f)(g) \mid g \in \overline{J(E(1))}^{w^*}_{\varepsilon,\lambda}\} \text{ on } A_\psi.$$

Consequently, we can obtain

$$\|\psi\|^{**} \geqslant |f(\psi)| \geqslant (\mathrm{Re}f)(\psi) > \bigvee\{(\mathrm{Re}f)(g) \mid g \in \overline{J(E(1))}^{w^*}_{\varepsilon,\lambda}\}$$

$$=\bigvee\{|f(g)| \mid g \in \overline{J(E(1))}^{w^*}_{\varepsilon,\lambda}\} \geqslant \bigvee\{|f(y)| \mid y \in E(1)\} = \|f\|^* \text{ on } A_\psi,$$

namely $\|\psi\|^{**}>1$ on A_ψ. □

Definition 4.4 ([5]). *Let E be a left module over the algebra $L^0(\mathcal{F},K)$ and G a subset of E. The set of countable concatenations $\Sigma_{n\geqslant 1}\tilde{I}_{A_n}x_n$ with $x_n \in G$ for each $n \in N$ is called the countable concatenation hull of G, denoted by $H_{cc}(G)$.*

Proof of Theorem 1.1. Denote $H_{cc}(E)$ by E_{cc} and define $\|\cdot\|_{cc}: E_{cc}\to L^0_+$ by $\|x\|_{cc}=\sum_{n\leqslant 1}\tilde{I}_{A_n}\cdot\|x_n\|$ for any $x=\sum_{n\leqslant 1}\tilde{I}_{A_n}\cdot x_n$ in E_{cc}, where $\{A_n\mid n\in N\}$ is a countable partition of Ω to $\mathcal{F}$ and $x_n\in E$ for any $n\in N$. It is easy to see that $E^{**}_{cc}=E^{**}$. By Theorem 4.1, we can obtain that $\overline{J(E_{cc}(1))}^{w^*}_{\varepsilon,\lambda}=E^{**}(1)$. Since $J(E(1))$ is dense in $J(E_{cc}(1))$ with respect to the $(\varepsilon,\lambda)-$topology which is induced by $\|\cdot\|_{cc}$ and stronger than $\sigma_{\varepsilon,\lambda}(E^{**},E^*)$, our desired result follows from the fact that the $(\varepsilon,\lambda)-$topology is stronger than $\sigma_{\varepsilon,\lambda}(E^{**},E^*)$. $\square$

Proof of Theorem 1.2. It follows immediately from Theorem 1.1 and Lemma 4.1. $\square$

The following example shows that $J(E(1))$ may be not dense in $E^{**}(1)$ under $\sigma_c(E^{**},E^*)$ if E has not the countable concatenation property.

Example 4.1. *Let $\Omega=\{1,2,3,\cdots\}$, $\mathcal{F}=2^{\Omega}$, $\bar{P}:\mathcal{F}\to R$ such that $\bar{P}(\Lambda)=$ the number of points in Λ if Λ is any finite subset in Ω and $\bar{P}(\Lambda)=\infty$ otherwise and $P:\mathcal{F}\to[0,1]$ such that $P(\Lambda)=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\bar{P}(\Lambda\cap\{n\})}{P(\{n\})}$ for each subset Λ of Ω, then $(\Omega,\mathcal{F},P)$ is a probability space. Let $(E,\|\cdot\|)=(L^0(\mathcal{F},K),|\cdot|)$ and $F=\{\varphi\in E\mid$ there is a positive integer n_φ such that $\varphi(k)=0,\ \forall k\geqslant n_\varphi\}$, then it is clear that F is an L^0-submodule of E and $(F,|\cdot|)$ is an RN module over K with the base $(\Omega,\mathcal{F},P)$. Let $F(1)=\{x\in F\mid |x|\leqslant 1\}$, then it is easy to check that $F(1)$ is a closed subset in both $(E,|\cdot|)$ and $(F,|\cdot|)$ under $\mathcal{T}_c$ induced by $|\cdot|$. Hence, we have that $F(1)$ is not dense in $E(1)$ under $\mathcal{T}_c$. Furthermore, it is clear that $(F^*,\|\cdot\|^*)=(F^{**},\|\cdot\|^{**})=(E,|\cdot|)$ and $\sigma_c(F^{**},F^*)$ is also the locally L^0- convex topology induced by $|\cdot|$. Consequently, $J(F(1))$ is not dense in $F^{**}(1)$ under $\sigma_c(F^{**},F^*)$.*

Acknowledgements The authors would like to thank our supervisor Professor Guo TieXin for some helpful and critical suggestions which considerably improve the readability of this paper.

References

[1] Guo T X. Survey of recent developments of random metric theory and its applications in China (I), Acta Anal. Funct. Appl. **3**(1): 129–158(2001)

[2] Guo T X. The theory of probabilistic metric spaces with applications to random functional analysis, Master's thesis, Xi'an: Xi'an Jiaotong University, 1989

[3] Breckner W.W, Scheiber E, A Hahn-Banach extension theorem for linear mappings into ordered modules, Mathematica, **19**(42): 13C27(1977)

[4] Vuza D, The Hahn-Banach theorem for modules over ordered rings, Rev. Roumaine Math. Pures Appl. **9**(27): 989C995(1982)

[5] Guo T X. Relations between some basic results derived from two kinds of topologies for a random locally convex module. J Funct Anal, **258**: 3024–3047(2010)

[6] Dunford N, Schwartz J T. Linear Operators. London: interscience, 1957

[7] Filipović D, Kupper M, Vogelpoth N. Separation and duality in locally L^0-convex modules. J Funct Anal. **256**: 3996-4029(2009)

[8] Guo T X, Xiao H X, Chen X X. A basic strict separation theorem in random locally convex modules, Nonlinear Anal Ser A. **71**: 3794-3804(2009)

[9] Guo T X. Survey of recent developments of random metric theory and its applications in China (II), Acta Anal. Funct. Appl. **3**(3): 208–230(2001)

[10] Guo T X, Chen X X. Random duality. Sci China Ser A, **52**: 2084–2098(2009)

[11] Guo T X. Recent progress in random metric theory and its applications to conditional risk measures, accepted for publication to Sci China Ser A (see also arXiv:1006.0697v10)

[12] Guo T X. Module homomorphisms on random normed modules. Chin Northeast Math J, **12**: 102–114(1996)

[13] Guo T X, Zhu L H. A characterization of continuous module homomorphisms on random seminormed modules and its applications, Acta Math. Sinica English Ser, **19**(1): 201–208(2003)

[14] Guo T X. Some basic theories of random normed linear spaces and random inner product spaces, Acta Anal Funct Appl, **1**(2): 160– 184(1999)

[15] Guo T X. The relation of Banach-Alaoglu theorem and Banach-Bourbaki-Kakutani-Smulian theorem in complete random normed modules to stratication structure, Sci China SerA, **51**: 1651–1663(2008)